

CENTERS OF INSCRIBED CIRCLES IN TRIANGULAR ORBITS OF AN ELLIPTIC BILLIARD

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ABSTRACT. The locus of centers of inscribed circles in triangles, the 3-periodic orbits of an elliptic billiard, is also an ellipse. In this work we obtain the canonical equation of this ellipse, complementing the previous results obtained by O. Romaskevich in [7].

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1. INTRODUCTION

Let \mathcal{E} be an ellipse and a point $p_1 \in \mathcal{E}$. Consider the triangle $\Delta(p_1) = \{p_1, p_2, p_3\}$ inscribed in \mathcal{E} that defines the 3-periodic of the billiard (the normal vector of the ellipse at the p_i is a bisectrix of $\Delta(p_1)$), associated to \mathcal{E} . See Fig. 1.

Next consider the geometric locus $\mathcal{E}_c = \{C(p_1), p_1 \in \mathcal{E}\}$, where $C(p_1)$ is defined as the center of the inscribed circle in the triangle $\Delta(p_1)$.

It is well known, by Poncelet's Theorem, that the sides of the family of triangles $\Delta(p_1)$ are tangent to a smaller ellipse \mathcal{E}_1 , confocal to the ellipse \mathcal{E} . See [8] and Fig. 2.

The main goal of this work is to obtain, using only techniques of real analytic and differential geometry, that \mathcal{E}_c is an ellipse. We present its canonical algebraic equation and also a parametrization which is a triple covering. Other geometrical related facts are also considered.

The fact that \mathcal{E}_c is an ellipse were established by O. Romaskevich, see [7], using techniques of complex algebraic geometry. See section 1 of the mentioned paper. Also we observe that in the work [7] no information about the shape of \mathcal{E}_c was achieved.

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Our strategy is to obtain the parametrization of the geometric locus \mathcal{E}_c and show that is a parametrized curve having positive constant affine curvature (therefore is an ellipse). This is the essence of Proposition 5. In Theorem 1 we describe explicitly the axes of ellipse \mathcal{E}_c and its canonical equation. Also it is worth to mention that the two ellipses \mathcal{E}_1 and \mathcal{E}_c are similar (the axes are proportional).

We make use of algebraic computational manipulators to perform the calculations. They are long in general, but can be checked by hand.

2. PRELIMINARIES

Consider an ellipse \mathcal{E} given implicitly by $h(x, y) = x^2/a^2 + y^2/b^2 - 1 = 0$. To fix the arguments it will be supposed that $a > b > 0$ and so the foci of \mathcal{E} are given by $(\pm\sqrt{a^2 - b^2}, 0) = (\pm c, 0)$.

Given a point $p_1 = (x_1, y_1) \in \mathcal{E}$ we have that $T_1 = (-\frac{y_1}{b^2}, \frac{x_1}{a^2})$ and $N_1 = (-\frac{x_1}{a^2}, -\frac{y_1}{b^2})$ is a positive orthogonal frame (basis) of \mathbb{R}^2 .

Proposition 1. *For any point $p_1 = (x_1, y_1) \in \mathcal{E}$ there exists a single triangle $\Delta(p_1) = \{p_1, p_2, p_3\}$ inscribed in \mathcal{E} such that $\Delta(p_1)$ is a billiard (3-periodic orbit) inscribed in \mathcal{E} . See Fig. 1.*

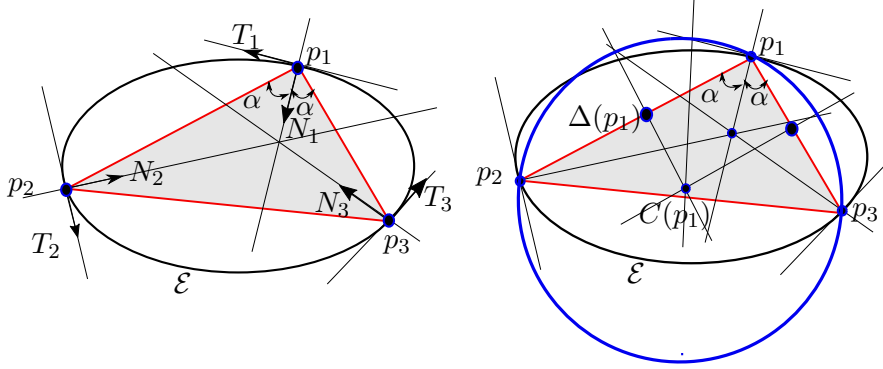


FIGURE 1. Triangular billiard orbit inscribed in the ellipse \mathcal{E} (left) and center $C(p_1)$ of the inscribed circle in the triangle $\Delta(p_1)$ (right).

Proof. The proof consists in the calculations of angles of incidence and reflection that the triangle $\Delta = \{p_1, p_2, p_3\}$ (the 3-periodic orbit), positively

oriented, makes at its apexes with the normal vector of the ellipse. See Fig. 1.

Let α be the angle of incidence and reflection at the point p_1 . So, we have defined two directions $d_{12} = \sin \alpha T_1 + \cos \alpha N_1$ and $d_{13} = -\sin \alpha T_1 + \cos \alpha N_1$.

Evaluating the intersection of the straight lines $p_1 + td_{12}$ and $p_1 + td_{13}$ with the ellipse \mathcal{E} we obtain the points p_2 and p_3 , defining a triangle $\Delta = \{p_1, p_2, p_3\}$ inscribed in the ellipse.

We have that $p_2 = (x_2, y_2)$, where $x_2 = \frac{p_{2x}}{q_2}$ and $y_2 = \frac{p_{2y}}{q_2}$ are given by:

$$\begin{aligned}
 p_{2x} &= -b^4 ((a^2 + b^2) \cos^2 t - a^2) x_1^3 - 2a^6 \cos t \sin t y_1^3 \\
 &\quad + a^4 ((a^2 - 3b^2) \cos^2 t + b^2) x_1 y_1^2 - 2a^4 b^2 \cos t \sin t x_1^2 y_1 \\
 p_{2y} &= 2b^6 \cos t \sin t x_1^3 - a^4 ((a^2 + b^2) \cos^2 t - b^2) y_1^3 \\
 &\quad + 2a^2 b^4 \cos t \sin t x_1 y_1^2 + b^4 ((b^2 - 3a^2) \cos^2 t + a^2) x_1^2 y_1 \\
 q_2 &= b^4 (a^2 - (a^2 - b^2) \cos^2 t) x_1^2 + a^4 (b^2 + (a^2 - b^2) \cos^2 t) y_1^2 \\
 &\quad - 2a^2 b^2 (a^2 - b^2) \cos t \sin t x_1 y_1
 \end{aligned} \tag{1}$$

Also, we have that $p_3 = (x_3, y_3)$, where $x_3 = \frac{p_{3x}}{q_3}$ and $y_3 = \frac{p_{3y}}{q_3}$ are given by:

$$\begin{aligned}
 p_{3x} &= b^4 (a^2 - (b^2 + a^2)) \cos^2 t x_1^3 + 2 \cos t a^6 \sin t y_1^3 \\
 &\quad + a^4 (\cos^2 t (a^2 - 3b^2) + b^2) x_1 y_1^2 + 2a^4 b^2 \cos t \sin t x_1^2 y_1 \\
 p_{3y} &= -2 \cos t b^6 \sin t x_1^3 + a^4 (b^2 - (b^2 + a^2) \cos^2 t) y_1^3 \\
 &\quad - 2a^2 b^4 \cos t \sin t x_1 y_1^2 + b^4 (a^2 + (b^2 - 3a^2) (\cos t)^2) x_1^2 y_1 \\
 q_3 &= b^4 (a^2 - (a^2 - b^2) \cos^2 t) x_1^2 + a^4 (b^2 + (a^2 - b^2) \cos^2 t) y_1^2 \\
 &\quad + 2a^2 b^2 (a^2 - b^2) \cos t \sin t x_1 y_1
 \end{aligned} \tag{2}$$

Therefore the segments $p_{12} = p_2 - p_1$ and $p_{13} = p_3 - p_1$ are, respectively, oriented by $d_{12} = \sin \alpha T_1 + \cos \alpha N_1$ and $d_{13} = -\sin \alpha T_1 + \cos \alpha N_1$.

Performing the calculations of angles of incidence and reflection at the points p_2 and p_3 and supposing that the triangle Δ is a 3-periodic orbit of the billiard we obtain the two equations,

$$\frac{\langle p_1 - p_2, N_2 \rangle}{|p_1 - p_2| |N_2|} = \frac{\langle p_3 - p_2, N_2 \rangle}{|p_3 - p_2| |N_2|}, \quad \frac{\langle p_1 - p_3, N_3 \rangle}{|p_1 - p_3| |N_3|} = \frac{\langle p_2 - p_3, N_3 \rangle}{|p_2 - p_3| |N_3|}.$$

A long and straightforward calculation, corroborated by algebraic symbolic computation, shows that the above equations have a common solution defined by:

$$(3) \quad \begin{aligned} & c^4 (b^4 x_1^2 + a^4 y_1^2)^2 \cos^4 \alpha + 2 a^4 b^4 (a^2 + b^2) (b^4 x_1^2 + a^4 y_1^2) \cos^2 \alpha - 3 a^8 b^8 = 0. \\ & c^4 |T_1|^4 \cos^4 \alpha + 2 (a^2 + b^2) |T_1|^2 \cos^2 \alpha - 3 = 0, \quad |T_1|^2 = \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}. \end{aligned}$$

The equation (3) has a single solution $\alpha \in (0, \frac{\pi}{2})$ and therefore defines uniquely the triangle $\Delta(p_1)$. This ends the proof. \square

By Poncelet's Theorem, see for example [1], [4], [5], and [8], the 3-periodic orbits given by the triangles $\Delta(p_1)$ are tangent to an ellipse \mathcal{E}_1 which is confocal with the ellipse \mathcal{E} .

Concretely we have the following result.

Proposition 2. *The smaller ellipse \mathcal{E}_1 of the 3-periodic billiard, confocal to the ellipse \mathcal{E} , is given by:*

$$(4) \quad \begin{aligned} h_1(x, y) &= \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 = 0, \\ a_1 &= -a(b^2 - \sqrt{b^4 - a^2 b^2 + a^4}) / (a^2 - b^2) > 0 \\ b_1 &= b(a^2 - \sqrt{b^4 - a^2 b^2 + a^4}) / (a^2 - b^2) > 0 \end{aligned}$$

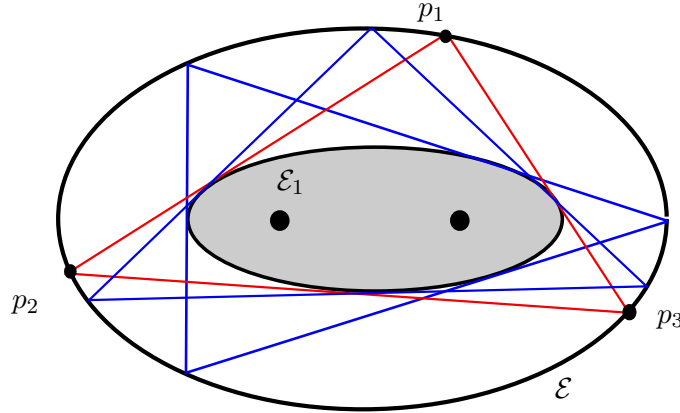


FIGURE 2. Confocal ellipses \mathcal{E} and \mathcal{E}_1 and a triangular orbit $\{p_1, p_2, p_3\}$.

Proof. It follows directly from Proposition 1, obtaining the triangles (3-periodic orbits) generated by the points $(\pm a, 0)$ and $(0, \pm b)$ and computing the smaller ellipse. \square

Lemma 1. *The center of the inscribed circle in the triangle $\Delta = \{p_1, p_2, p_3\}$, $p_i = (x_i, y_i)$, is given by (x_c, y_c) , where:*

$$(5) \quad \begin{aligned} x_c &= \frac{1}{2} \frac{(x_1^2 + y_1^2)(y_2 - y_3) + (x_2^2 + y_2^2)(y_3 - y_1) + (x_3^2 + y_3^2)(y_1 - y_2)}{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}, \\ y_c &= \frac{1}{2} \frac{(x_1^2 + y_1^2)(x_3 - x_2) + (x_2^2 + y_2^2)(x_1 - x_3) + (x_3^2 + y_3^2)(x_2 - x_1)}{y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1)} \end{aligned}$$

Proof. Direct calculation. We obtain the intersection between medial lines $(p_1 + p_2)/2 + t(p_2 - p_3)^\perp$ and $(p_1 + p_3)/2 + t(p_1 - p_3)^\perp$, bisectors of triangle $\Delta(p_1)$, where $(p_i - p_j)^\perp$ is a vector orthogonal to $p_i - p_j$. \square

Proposition 3. *Let $C(p_1)$ the center of inscribed circle in the triangle $\Delta(p_1)$ that defines the 3-periodic orbit of the billiard associated to the confocal ellipses \mathcal{E} and \mathcal{E}_1 . Then the locus $\{C(p_1), p \in \mathcal{E}\}$ is parametrized by (x_c, y_c) , where:*

$$(6) \quad \begin{aligned} x_c &= \frac{x_1}{a} \left(\frac{A^3 x_1^2 + a_{02} y_1^2}{A^2 x_1^2 + B^2 y_1^2} \right), \quad y_c = \frac{y_1}{b} \left(\frac{b_{20} x_1^2 + B^3 y_1^2}{A^2 x_1^2 + B^2 y_1^2} \right), \\ A &= \frac{a^2 - \sqrt{a^2 c^2 + b^4}}{2a}, \quad B = \frac{b^2 - \sqrt{a^2 c^2 + b^4}}{2b}, \\ a_{02}(a, b) &= \frac{a((3a^3 + ab^2)A - b^2 c^2)}{4b^2}, \quad b_{20}(a, b) = \frac{b((a^2 b + 3b^3)B + a^2 c^2)}{4a^2}. \end{aligned}$$

Proof. It follows directly from Lemma 1 and Proposition 1.

We observe that

$$\begin{aligned} x_c &= \frac{x_1 F_1}{2a^2 F_3} = \frac{x_1}{2a^2} \frac{a_{20}(a, b)x_1^2 + a_{02}(a, b)y_1^2}{p(a, b)x_1^2 + q(a, b)y_1^2} \\ y_c &= \frac{y_1 F_2}{2b^2 F_3} = \frac{y_1}{2b^2} \frac{b_{20}(a, b)x_1^2 + b_{02}(a, b)y_1^2}{p(a, b)x_1^2 + q(a, b)y_1^2} \\ a_{20}(a, b) &= b_{02}(b, a), \quad a_{02}(a, b) = b_{20}(b, a). \end{aligned}$$

where,

$$\begin{aligned}
F_1 &= (b^2 (4a^4 - a^2b^2 + b^4) A_2 - a^2b^2 (4a^4 - 3a^2b^2 + 3b^4)) x_1^2 \\
&\quad + a^4 ((3a^2 + b^2) A_2 - 3a^4 + a^2b^2 - 2b^4) y_1^2 \\
F_2 &= (b^4 (a^2 + 3b^2) A_2 - b^4 (2a^4 - a^2b^2 + 3b^4)) x_1^2 \\
(7) \quad &\quad + (a^2 (a^4 - a^2b^2 + 4b^4) A_2 - a^2b^2 (3a^4 - 3a^2b^2 + 4b^4)) y_1^2 \\
F_3 &= (2a^2b^2 A_2 - b^2 (2a^4 - a^2b^2 + b^4)) x_1^2 \\
&\quad + (2a^2b^2 A_2 - a^2 (2b^4 + a^4 - a^2b^2)) y_1^2 \\
A_2 &= \sqrt{a^2(a^2 - b^2) + b^4} = \sqrt{a^2c^2 + b^4}
\end{aligned}$$

Simplifying the above equation it follows the result. \square

In terms of the axes of ellipses \mathcal{E} and \mathcal{E}_1 we have the following proposition.

Proposition 4. *Let $p_1 = (x_1, y_1) \in \mathcal{E}$ and $C(p_1)$ the center of the inscribed circle in the triangle $\Delta(p_1)$ that defines the 3-periodic orbit of the billiard associated to the confocal ellipses \mathcal{E} and \mathcal{E}_1 . Then the locus $\{C(p_1) : p_1 \in \mathcal{E}\}$ is parametrized by (x_c, y_c) , where:*

$$\begin{aligned}
(8) \quad x_c &= \frac{x_1}{4} \frac{b^2 (b_1^2(b^4 - b^2a^2 + 4a^4) - b^6) x_1^2 + a^4 (b^2b_1^2 - b^4 + 3b_1^2a^2) y_1^2}{(b_1^2x_1^2 + a_1^2y_1^2)a^4b^2} \\
y_c &= \frac{y_1}{4} \frac{b^4 (a_1^2a^2 - a^4 + 3a_1^2b^2) x_1^2 + a^2 (a_1^2(a^4 - b^2a^2 + 4b^4) - a^6) y_1^2}{(b_1^2x_1^2 + a_1^2y_1^2)a^2b^4}
\end{aligned}$$

Proof. It follows from Propositions 2 and 3. \square

3. AFFINE CURVATURE OF PLANE CURVES

The affine group of transformations of the plane is generated by translations and linear isomorphisms.

As in the Euclidean case, it is natural to consider length and curvature of plane curves which are invariant by the special affine group.

Consider a regular plane curve $c(s) = (x(s), y(s))$ in the affine plane \mathbb{A}^2 with the canonical area form $\omega = dx \wedge dy$ and suppose that $[c'(s), c''(s)] = 1$. This is always possible when c is a convex curve, i.e., its Euclidean curvature is positive.

Differentiating this equation it follows that $[c'(s), c'''(s)] = 0$ and, as $c' \neq 0$, it is obtained.

$$c'''(s) + k_a(s)c'(s) = 0 \Rightarrow k_a = [c''(s), c'''(s)].$$

Above $[u, v] = u_1v_2 - u_2v_1$ is the determinant of matrix formed by the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

The function k_a is called the *affine curvature* of c . See [6] and [9].

With respect a parametrization $\gamma(u) = c(s(u))$ it follows that

$$\begin{aligned}\gamma' &= c_s s' \\ \gamma'' &= c_{ss} s'^2 + c_s s'' \\ \gamma''' &= c_{sss} s'^3 + 3c_{ss} s' s'' + c_s s'''\end{aligned}$$

where, $c_s = dc/ds = c'$, $c_{ss} = d^2c/ds^2 = c''$ and $c_{sss} = d^3c/ds^3 = c'''$.

Moreover,

$$\begin{aligned}s'^3 &= [\gamma', \gamma''], \\ 3s'^2 s'' &= [\gamma', \gamma'''] \\ 6s' s''^2 + 3s'^2 s''' &= [\gamma', \gamma''''] + [\gamma'', \gamma'''].\end{aligned}$$

Now, using that $k_a(s) = [c_{ss}, c_{sss}]$, we obtain, in terms of the parametrization $\gamma(u)$, that :

$$(9) \quad k_a(u) = \frac{4[\gamma'', \gamma'''] + [\gamma', \gamma'''']}{3[\gamma', \gamma'']^{\frac{5}{3}}} - \frac{5}{9} \frac{[\gamma', \gamma''']^2}{[\gamma', \gamma'']^{\frac{8}{3}}}.$$

Remark 1. The affine curvature of the ellipse \mathcal{E} is given by $1/(ab)^{\frac{2}{3}}$.

4. MAIN RESULTS

Proposition 5. The affine curvature k_a of the curve of centers $\{C(p_1) : p_1 \in \mathcal{E}\}$ of the inscribed circles in triangles of an elliptic billiard is constant and is given by:

$$(10) \quad k_a^3 = \frac{1}{A^2 B^2}, \quad A = \frac{a^2 - \sqrt{a^4 - a^2 b^2 + b^4}}{2a}, \quad B = \frac{b^2 - \sqrt{a^4 - a^2 b^2 + b^4}}{2b}.$$

Proof. A long calculation, confirmed by algebraic symbolic computation, gives that the affine curvature of $C(p_1)$ is constant and positive.

In fact, using the parametrization given by equation (7), making $x_1 = au$ and $y_1 = b\sqrt{1-u^2}$ and with the expression for the affine curvature given by equation (9), it follows that:

$$\begin{aligned}
k_a^3 &= -\frac{16a^2b^2U^3}{V^8} \\
U &= -36493a^6b^{18} - 6561a^{22}b^2 - 38546a^{20}b^4 - 36493a^{18}b^6 - 62918a^{16}b^8 \\
&\quad - 67282a^{14}b^{10} - 74444a^{12}b^{12} - 67282a^{10}b^{14} \\
&\quad - 62918a^8b^{16} - 38546a^4b^{20} - 6561a^2b^{22} - 13122a^{24} - 13122b^{24} \\
&\quad + 2A_2(b^2 + a^2)(81b^8 - 20b^6a^2 + 134b^4a^4 - 20b^2a^6 + 81a^8) \\
&\quad (81b^{12} + 20b^{10}a^2 + 119b^8a^4 + 72b^6a^6 + 119b^4a^8 + 20b^2a^{10} + 81a^{12}) \\
V &= (27a^8 + 20b^2a^6 + 34b^4a^4 + 20b^6a^2 + 27b^8)A_2 \\
&\quad - (b^2 + a^2)(27b^8 - 20b^6a^2 + 50b^4a^4 - 20b^2a^6 + 27a^8)
\end{aligned}$$

The above equation admits the factorization $k_a^3 = 1/(AB)^2$. This was confirmed by algebraic symbolic computation.

As the affine curvature k_a is constant and positive it follows that the geometric locus $\mathcal{E}_c = \{C(p_1), p_1 \in \mathcal{E}\}$ is an ellipse. \square

Theorem 1. *The locus $\mathcal{E}_c = \{C(p_1), p_1 \in \mathcal{E}\}$ is an ellipse with canonical equation $x^2/A^2 + y^2/B^2 = 1$, where*

$$(11) \quad A = \frac{a^2 - \sqrt{a^2c^2 + b^4}}{2a}, \quad B = \frac{b^2 - \sqrt{a^2c^2 + b^4}}{2b}.$$

The foci of \mathcal{E}_c are $(0, \pm \frac{c^3}{2ab})$. In relation to the axes a_1 and b_1 of the smaller ellipse \mathcal{E}_1 it follows that $4a^2b^2A^2 = c^4b_1^2$ and $4a^2b^2B^2 = c^4a_1^2$. In particular, $\frac{b_1}{a_1} = \frac{A}{|B|}$.

If $a > b \geq \frac{a}{4}\sqrt{-1 + \sqrt{33}}$ the ellipse $x^2/A^2 + y^2/B^2 = 1$ is contained in the region bounded by ellipse $x^2/a^2 + y^2/b^2 = 1$.

Proof. By Proposition 5 we have that the \mathcal{E}_c is an ellipse.

This ellipse contains the five points $P_i, (i = 1, \dots, 5)$, where $P_1 = \gamma(a, 0)$, $P_2 = \gamma(-a, 0)$, $P_3 = \gamma(0, b)$, $P_4 = \gamma(0, -b)$, $P_5 = \gamma(a/2, (\sqrt{3}/2)b)$.

Elementary calculation, solving a linear system of equation, although long, shows that the ellipse is as affirmed.

With the hypothesis that $a > b$, we have $B^2 > A^2$ and $C^2 = B^2 - A^2 = \frac{(a^2-b^2)^3}{4a^2b^2}$. Therefore the foci are given by $(0, \pm C)$, where $C = c^3/(2ab)$.

Another way to confirm the result is that, defining $H(x, y) = x^2/A^2 + y^2/B^2 - 1$, we obtain directly that $H(x_c(a \cos t, b \sin t), y_c(a \cos t, b \sin t)) = 0$.

The algebraic relations $4a^2b^2A^2 = b_1^2c^4$ and $4a^2b^2B^2 = a_1^2c^4$ follows directly from the definition of quantities involved.

The inequality $B \leq b$ is equivalent to the condition $b \geq \frac{a}{4} \sqrt{-1 + \sqrt{33}}$. This ends the proof. \square

Proposition 6. *The curve $\gamma(t) = (x_c(t), y_c(t)) = (x_c(x_1, y_1), y_c(x_1, y_1))$, where $\Gamma(t) = (x_1, y_1) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$, is a triple covering of the ellipse \mathcal{E}_c , i.e., defining $C : \mathcal{E} \rightarrow \mathcal{E}_c$ by $C(\Gamma(t)) = \gamma(t)$ it follows that $C^{-1}(p_c)$ always consists of three points for all $p_c \in \mathcal{E}_c$.*

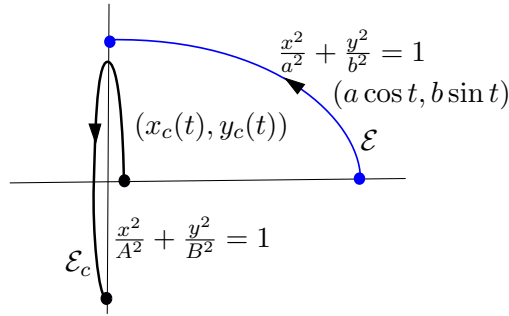


FIGURE 3. Sketch of the ellipses \mathcal{E} (curve Γ) and \mathcal{E}_c (curve γ) in the interval $[0, \pi/2]$.

Proof. Consider a rational function $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the type

$$F(x, y) = \left(x \frac{ax^2 + by^2}{px^2 + qy^2}, y \frac{cx^2 + dy^2}{px^2 + qy^2} \right)$$

We have that,

$$\det(Jac(F)) = \frac{acx^4 + (3ad - bc)x^2y^2 + bdy^4}{(px^2 + qy^2)^2}.$$

Applying the above result to the parametrization of ellipse \mathcal{E}_c given by equation (6) it follows that:

$$\begin{aligned} \det(Jac(F)) &= \frac{P}{Q^2} \\ P &= -(b^4 x_1^2 + a^4 y_1^2)(\alpha x_1^2 + \beta y_1^2) \\ \alpha &= b^2 (5 a^2 b^6 + 3 b^2 a^6 + 9 a^4 b^4 + 12 a^8 + 3 b^8) \sqrt{a^4 - a^2 b^2 + b^4} \\ &\quad - b^2 (12 a^{10} + 4 a^2 b^8 + 3 b^{10} - 3 a^8 b^2 + 12 b^4 a^6 + 4 a^4 b^6) \\ \beta &= a^2 (12 b^8 + 5 b^2 a^6 + 3 a^8 + 9 a^4 b^4 + 3 a^2 b^6) \sqrt{a^4 - a^2 b^2 + b^4} \\ &\quad - a^2 (3 a^{10} + 4 a^8 b^2 - 3 a^2 b^8 + 4 b^4 a^6 + 12 a^4 b^6 + 12 b^{10}) \\ Q &= 2ab^3 \left(-b^4 + a^2 b^2 - 2 a^4 + 2 a^2 \sqrt{a^4 - a^2 b^2 + b^4} \right) x_1^2 \\ &\quad + 2a^3 b \left(-a^4 + a^2 b^2 - 2 b^4 + 2 b^2 \sqrt{a^4 - a^2 b^2 + b^4} \right) y_1^2. \end{aligned}$$

By algebraic manipulation we obtain that $\alpha < 0$ and $\beta < 0$ and therefore $\det(Jac(F)) > 0$ for all $(x_1, y_1) \neq (0, 0)$.

Moreover, we have that the function $x_c(x_1, y_1) = x_c(t)$ has six zeros, which are given by $x_1 = 0$ and $y_1 = \pm k_1(a, b)x_1$. The same for the function $y_c(x_1, y_1) = y_c(t)$ that vanishes at $y_1 = 0$ and $y_1 = \pm k_2(a, b)x_1$. Therefore, each coordinate function has 6 zeros in interval $[0, 2\pi)$.

A typical example of this behavior is shown in Fig. 4 below.

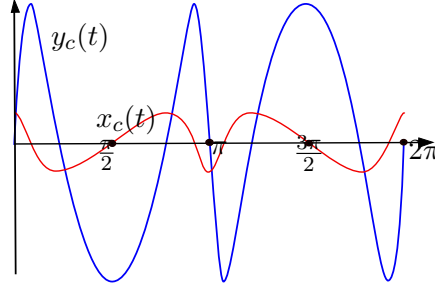


FIGURE 4. Graphic of coordinate functions x_c and y_c of the ellipse \mathcal{E}_c .

Then $\gamma(t) = (x_c(a \cos t, b \sin t), y_c(a \cos t, b \sin t))$ is a triple covering of the ellipse \mathcal{E}_c . In fact, $C^{-1}(C(p_1)) = \{p_1, p_2, p_3\} = \Delta(p_1)$.

The Fig. 3 shows the trace of $\Gamma(t) = (a \cos t, b \sin t)$ and $\gamma(t)$ in the interval $[0, \pi/2]$. This ends the proof. \square

Proposition 7. *The function*

$$F(a, b) = \left(\frac{-a \left(b^2 - \sqrt{b^4 - a^2 b^2 + a^4} \right)}{a^2 - b^2}, \frac{b \left(a^2 - \sqrt{b^4 - a^2 b^2 + a^4} \right)}{a^2 - b^2} \right)$$

is a global diffeomorphism of the open region $R = \{(a, b) : a > 0, b > 0 \text{ and } a > b\}$.

Proof. Direct calculation shows that $\det(\text{Jac}(F)) > 0$ in the region R and that $F(R) \subset R$. Therefore, by Inverse Function Theorem, F is a local diffeomorphism. To obtain the result we observe that the equation $F(a, b) = (x, y)$ is equivalent to the system $x^2 - y^2 = a^2 - b^2$, $bx + ay = ab$. With the hypothesis $x > y$, the two hyperbolas in the plane ab have only one transversal point of intersection in the region R . In fact, writing $c^2 = x^2 - y^2 > 0$ we have that $b = ay/(a - x)$ and a satisfies a polynomial equation $p(a) = a^4 - 2a^3x + 2c^2ax - c^2x^2 = 0$ which has only one positive solution.

This follows from the fact the discriminant of equation above is $-432x^4c^4(-x^2 + c^2)^2 < 0$ (see [2]) and therefore the quartic polynomial equation $p(a) = 0$ has only two real roots and in our case the positive one is bigger than x . \square

5. CONCLUSION

In this work, inspired by the paper of O. Romaskevich [7], we established in Theorem 1 that the centers of circles inscribed in triangular orbits of an elliptic billiard is also an ellipse \mathcal{E}_c .

The main contribution of this work it was the calculation of the canonical equation of ellipse \mathcal{E}_c . Also it is useful to observe that the axes of ellipse \mathcal{E}_c are proportional to the axes of a smaller ellipse \mathcal{E}_1 (confocal to the ellipse \mathcal{E}) associated to the 3-periodic billiard. Also it was established in Proposition 6 that the parametrized ellipse \mathcal{E} (curve Γ) is a triple covering of \mathcal{E}_c (curve γ).

Finally we observe that the geometric locus of barycenters and orthocenters of the triangles $\Delta(p_1)$ (the 3-periodic orbits associated to an elliptic billiard) are also ellipses. See [3].

REFERENCES

- [1] DRAGOVIĆ, VLADIMIR and RADNOVIĆ, MILENA, *Bicentennial of the great Poncelet theorem (1813–2013): current advances*, Bull. Amer. Math. Soc. (N.S.), **51**, (2014), 373–445.
- [2] W. S. BURNSIDE and A. W. PANTON, *The Theory of Equations*, vol. 2, New York, Dover Publications, Inc. (1912).
- [3] R. GARCIA, *Elliptic billiards and ellipses associated to the 3-periodic orbits*. In preparation.
- [4] G. GLAESER, H. STACHEL and B. ODEHNAL, *The Universe of Conics, From the ancient Greeks to 21st century developments*. Springer Verlag (2016).
- [5] P. GRIFFITHS and J. HARRIS, *On Cayley’s explicit solution to Poncelet’s porism*, Enseign. Math. (2) **24** (1978), no. 1-2, 31-40.
- [6] K. NOMIZU and T. SASAKI, *Affine differential geometry. Geometry of affine immersions*. Cambridge Tracts in Mathematics, 111. Cambridge Univ. Press, Cambridge, 1994.
- [7] O. ROMASKEVICH, *On the incenters of triangular orbits on elliptic billiards*, Enseign. Math. **60** (2014), no. 3-4, 247-255.
- [8] S. TABACHNIKOV, *Geometry and Billiards*, American Mathematical Society, Providence, RI; (2005), xii+176.
- [9] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, vol.III, Publish of Perish Berkeley, (1979).

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